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Integrable models from twisted half-loop algebras

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Abstract
This paper is devoted to the construction of new integrable quantum-mechanical models based on certain subalgebras of the half-loop algebra of $\mathfrak{gl}_N$. Various results about these subalgebras are proven by presenting them in the notation of the St Petersburg school. These results are then used to demonstrate the integrability, and find the symmetries, of two types of physical system: twisted Gaudin magnets and Calogero-type models of particles on several half lines meeting at a point.

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1. Introduction

This paper has two motivations. On the one hand, we are interested in physical models of particles on a number of half lines joined at a central point. Such systems, for free particles, have been treated in, for example, [1, 2]. Here we would like to consider interacting models, to establish that integrable examples of such models exist, and to find their symmetries. We shall work out explicitly two examples: the Gaudin model [3] and the Calogero model [4]. Both have numerous applications in physics and in mathematics. For example, the reduced BCS model for conventional superconductivity can be diagonalized in an algebraic way [5] using the Gaudin model. Other, more recent, applications of the Gaudin model in quantum many-body physics can be found for example in the reviews of [6, 7]. Besides being of intrinsic interest due to its exact solvability, the Calogero model plays a role in the study of two-dimensional Yang–Mills theory [8], the quantum-Hall effect [9] and fractional statistics [10].

Our second motivation is algebraic. The notation of the St Petersburg School [11] is a powerful tool when working with the Yangian [12] of $\mathfrak{gl}_N$ and its subalgebras: the reflection algebras [13, 14] and twisted Yangians [15]. These are quantum algebras, but the construction has a classical limit in which the quantum R-matrix and Yang–Baxter equation are replaced by their classical counterparts (see for example [16]). The classical limit of the Yangian is the half-loop algebra, and the limits of the reflection algebras and twisted Yangian are subalgebras of this half-loop algebra defined by automorphisms of order 2. But there also exist, at least in the classical case, other subalgebras of the half-loop algebra, defined by automorphisms of higher finite order. We wish to study these subalgebras using classical r-matrix techniques.

It is well known that the half-loop algebras associated with Lie algebras are crucial in the study of Gaudin models and Calogero models. These algebras provide, in the former case, a systematic way to construct the model (see e.g [17]) and, in the latter case, the symmetry algebras of the system [18–20]. In both cases, they allow one to prove the integrability of the model. We shall find similar connections in the cases studied in this paper. Indeed, we shall see below that the order n subalgebras of the half-loop algebra appear naturally in the description of models on n half lines.

This paper is structured as follows. We begin with a brief review of the half-loop algebra of $\mathfrak{gl}_N$ and its subalgebras associated with automorphisms of order n. We make use of the notation of the St Petersburg school to find Abelian subalgebras. In the subsequent sections, these algebraic results are shown to provide new quantum-integrable models and demonstrate their symmetries: section 3 discusses 'twisted' Gaudin magnets, and section 4 introduces Calogero-type models on n half lines joined to a central point. We end with some conclusions and a short discussion of classical counterparts of these results.

2. Half-loop algebra and subalgebras

2.1. St Petersburg notation and half-loop algebra

The half-loop algebra $\mathcal{H}_N$ based on $\mathfrak{gl}_N$ is the complex associative unital algebra with the following set of generators: $\{t_{ij}^{(\alpha)} | 1 \leqslant i, j \leqslant N, \alpha \in \mathbb{Z}_{\geqslant 0}\}$, subject to the defining relations

$$\left[t_{ij}^{(\alpha)}, t_{kl}^{(\beta)}\right] = \delta_{jk}\, t_{il}^{(\alpha+\beta)} - \delta_{il}\, t_{kj}^{(\alpha+\beta)} \tag{2.1}$$

for $\alpha, \beta \geqslant 0$ and $1 \leqslant i, j, k, l \leqslant N$. It is isomorphic to the algebra $\mathfrak{gl}_N[z]$ of polynomials in an indeterminate z with coefficients in $\mathfrak{gl}_N$, with the generators identified as follows:

$$t_{ij}^{(\alpha)} = e_{ij} z^{\alpha}, \tag{2.2}$$

where e_{ij} are the generators of $\mathfrak{gl}_N$, satisfying the commutation relations

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj}. \tag{2.3}$$

It will simplify our computations to introduce the notation of the St Petersburg school: let E_{ij} be the $N \times N$ matrix with a 1 in the ijth slot and zeros elsewhere. These are the generators of $\mathfrak{gl}_N$ in the fundamental representation. Let us now gather the generators of $\mathcal{H}_N$ in the matrix

$$T(u) = \sum_{i,j=1}^{N} E_{ij} \otimes \sum_{\alpha \geqslant 0} \frac{t_{ji}^{(\alpha)}}{u^{\alpha+1}} = \sum_{i,j=1}^{N} E_{ij} \otimes T_{ji}(u) = \sum_{\alpha \geqslant 0} \frac{T^{(\alpha)}}{u^{\alpha+1}}, \tag{2.4}$$

where $T^{(\alpha)} = \sum_{i,j=1}^{N} E_{ij} \otimes t_{ji}^{(\alpha)}$ ($\alpha \geqslant 0$) and u is a formal parameter called the spectral parameter. Note the flip of the indices between E_{ij} and t_{ji}, which will prove convenient later. The algebraic object $T(u)$ is an element of $\mathrm{Mat}_{N\times N} \otimes \mathcal{H}_N[[u^{-1}]]$, and as usual we refer to $\mathrm{Mat}_{N\times N}$ as the auxiliary space and $\mathcal{H}_N$ as the algebraic space. In what follows, we shall require several copies of both spaces. We use letter $a, b, \ldots$ from the start of the alphabet to refer to copies of the auxiliary space and numerals $1, 2, \ldots$ for copies of the algebraic space. Let us also introduce

$$r_{ab}(u) = \frac{P_{ab}}{u}, \tag{2.5}$$

where $P_{ab} = \sum_{i,j=1}^{N} E_{ij} \otimes E_{ji}$ is the permutation operator between two auxiliary spaces: the letters a and b stand respectively for the first and the second spaces. By the definition, it satisfies $P_{ab} v \otimes w = w \otimes v$ $(v, w \in \mathbb{C}^N)$. The matrix $r_{ab}(u)$, usually called the classical R-matrix (see for example [16]), satisfies the classical Yang–Baxter equation

$$[r_{ab}(u_a - u_b), r_{ac}(u_a - u_b)] + [r_{ab}(u_a - u_b), r_{bc}(u_b - u_c)] \\ + [r_{ac}(u_a - u_c), r_{bc}(u_b - u_c)] = 0 \tag{2.6}$$

and allows us to encode the half-loop algebra defining relations (2.1) in the simple equation

$$[T_a(u), T_b(v)] = [T_a(u) + T_b(v), r_{ab}(u - v)]. \tag{2.7}$$

This form of commutation relations can be obtained easily by taking the classical limit of the presentation of the Yangian of $\mathfrak{gl}_N$ [12] introduced by Faddeev, Reshetikhin and Takhtajan of St Petersburg [11]. By taking the trace in the space a in (2.7), it is straightforward to show that the coefficients of the series $t(u) = \mathrm{tr}_a T_a(u)$ are central. The quotient of the algebra $\mathcal{H}_N$ by the relation $t(u) = 0$ is isomorphic to the polynomial algebra $\mathfrak{sl}_N[z]$.

Identification (2.2) between the generators of $\mathcal{H}_N$ and $\mathfrak{gl}_N[z]$ now reads

$$T_a(u) = \frac{\mathbb{P}_{a1}}{u - z}, \tag{2.8}$$

where

$$\mathbb{P}_{a1} = \sum_{i,j=1}^{N} E_{ij} \otimes e_{ji}, \tag{2.9}$$

and $\frac{1}{u-z}$ is to be understood as the formal series $\sum_{\alpha \geqslant 0} \frac{z^\alpha}{u^{\alpha+1}}$. Note the similarity between relations (2.5) and (2.8): the only differences are that the second auxiliary space (denoted b) in (2.5) is replaced by an algebraic space (denoted 1) and that the spectral parameter is shifted by z. In fact, there exists a more general solution of relations (2.7) in the L-fold tensor product of $\mathfrak{gl}_N[z]$,

$$T_a(u) = \sum_{\ell=1}^{L} \frac{\mathbb{P}_{a\ell}}{u - z_\ell}. \tag{2.10}$$

From now on, we work in the enveloping algebra $\mathcal{U}(\mathcal{H}_N)$ in which, for example, the product $T_a(u)^2$ makes sense.

2.2. *The inner-twisted algebras*

Let σ be an inner automorphism of $\mathfrak{gl}_N$ of order n. One way to define σ is by its action on matrices $X \in \mathrm{Mat}_{N\times N}$ in the fundamental representation:

$$\sigma : X \mapsto G^{-1} X G, \tag{2.11}$$

where $G \in \mathrm{Mat}_{N\times N}$ satisfies $G^n = 1$; the action of σ on the abstract algebra $\mathfrak{gl}_N$ is then given by $\sigma : e_{ij} \mapsto G_{jq}(G^{-1})_{pi} e_{pq}$, or, in the notation of the previous section,

$$\sigma : \mathbb{P}_{a1} \mapsto G_a \mathbb{P}_{a1} G_a^{-1}. \tag{2.12}$$

The eigenvalues of σ are the nth roots of unity $\tau^k := \exp 2\pi \mathrm{i} k/n$, and for each $k \in \mathbb{Z}_n := \frac{\mathbb{Z}}{n\mathbb{Z}}$ the map

$$P_k = \frac{1}{n} \sum_{j \in \mathbb{Z}_n} \tau^{-jk} \sigma^j \tag{2.13}$$

is the projector onto the τ^k eigenspace:

$$\sigma P_k = \tau^k P_k, \qquad P_k P_j = \delta_{jk} P_k. \tag{2.14}$$

Since $1 = P_0 + P_1 + \cdots + P_{n-1}$, $\mathfrak{gl}_N$ decomposes into the direct sum of eigenspaces of σ. This decomposition respects the Lie bracket, in the sense that if $\sigma e = \tau^k e$ and $\sigma f = \tau^l f$ then

$$\sigma[e, f] = [\sigma e, \sigma f] = \tau^{k+l}[e, f], \tag{2.15}$$

and is said to be a $\mathbb{Z}_n$-gradation of $\mathfrak{gl}_N$.

By a change of basis we can take

$$G = \mathrm{diag}(\underbrace{1, \ldots, 1}_{N_0}, \underbrace{\tau, \ldots, \tau}_{N_1}, \ldots, \underbrace{\tau^{n-1}, \ldots, \tau^{n-1}}_{N_{n-1}}), \tag{2.16}$$

where $N_0 + N_1 + \cdots + N_{n-1} = N$. Note that the +1-eigenspace of σ is the Lie subalgebra $\mathfrak{gl}_{N_0} \oplus \mathfrak{gl}_{N_1} \oplus \ldots \oplus \mathfrak{gl}_{N_{n-1}}$.

Let us define

$$\mathfrak{gl}_N[z]^\sigma = \{A(z) \in \mathfrak{gl}_N[z] \mid \sigma A(z) = A(\tau z)\}, \tag{2.17}$$

that is, $\mathfrak{gl}_N[z]^\sigma$ is the subalgebra of $\mathfrak{gl}_N[z]$ in which each element of degree k is also in the τ^k-eigenspace of σ. There is a surjective projection map $\mathfrak{gl}_N[z] \to \mathfrak{gl}_N[z]^\sigma$, defined by $ez^k \mapsto P_k e z^k$. In view of (2.12), this sends

$$T(u) \mapsto \sum_{j\in\mathbb{Z}_n} \tau^j G^j T(u\tau^j) G^{-j} =: B(u), \tag{2.18}$$

which defines the formal series $B(u)$ whose expansion

$$B(u) = \frac{1}{u} B^{(0)} + \frac{1}{u^2} B^{(1)} + \cdots, \tag{2.19}$$

contains by construction a complete set of generators of $\mathfrak{gl}_N[z]^\sigma$.

Lemma 2.1. *$B(u)$ obeys*

$$[B_a(u), B_b(v)] = \sum_{k\in\mathbb{Z}_n} \left[\tau^k B_a(u) + B_b(v), \frac{G_a^{-k} P_{ab} G_a^k}{u - \tau^k v} \right] \tag{2.20}$$

and has the property that for all $k \in \mathbb{Z}_n$

$$B(u) = \tau^k G^k B(u\tau^k) G^{-k}. \tag{2.21}$$

Proof. The first of these is true by virtue of (2.7), while the second follows immediately from definition (2.18). □

The coefficients in the expansion of $b(u) = \mathrm{tr}\, B(u)$ are central in $\mathfrak{gl}_N[z]^\sigma$, as may be seen by taking the trace in space a or b in (2.20). But there exist also other Abelian subalgebras in $\mathcal{U}(\mathfrak{gl}_N[z]^\sigma)$ as follows.

Proposition 2.2. *The coefficients in the expansion of $b'(u) = \mathrm{tr}\, B(u)^2$ are mutually commuting, or equivalently*

$$[b'(u), b'(v)] = 0, \tag{2.22}$$

for all values of u and v. Moreover, they commute with the generators of $\mathfrak{gl}_N[z]^\sigma$ of degree zero:

$$[B^{(0)}, b'(u)] = 0. \tag{2.23}$$

The algebraic elements in $B^{(0)}$ generate $\mathfrak{gl}_{N_0} \oplus \mathfrak{gl}_{N_1} \oplus \cdots \oplus \mathfrak{gl}_{N_{n-1}}$.

Proof. The details of the proof are given in appendix A. □

In particular, we recover (for $n = 1$) the fact that $\mathrm{tr}\, T(u)^2$ commute and (for $n = 2$) the results of Hikami [19] concerning the classical limit of the reflection algebra.

In sections 3 and 4, we will apply this purely algebraic result to find new integrable models.

2.3. Outer automorphisms

In the previous section, we focused on inner automorphisms. Now, we show how to modify the construction to study outer automorphisms. Modulo inner automorphisms, the only outer automorphism of $\mathfrak{gl}_N$, are generalized transposition, which have order 2.

Let K be a real invertible $N \times N$ matrix satisfying $K^t = \eta K$ with $\eta = \pm 1$ (for $\eta = -1$, N must be even), and define an outer automorphism $\mathcal{T}$ by $e_{ij} \mapsto K_{jp}(K^{-1})_{qi} e_{pq}$, or equivalently

$$\mathcal{T} : \mathbb{P}_{a1} \mapsto \mathbb{P}_{a1}^{\mathcal{T}_a} = K_a \mathbb{P}_{a1}^{t_a} K_a^{-1} =: \mathbb{Q}_{a1}, \tag{2.24}$$

where t_a is matrix transposition in the space a. The eigenvalues of $\mathcal{T}$ are ± 1 and, as before, the decomposition of $\mathfrak{gl}_N$ into the direct sum of eigenspaces of $\mathcal{T}$ defines a $\mathbb{Z}_2$-gradation.

One may introduce the $N \times N$ matrices

$$\mathcal{G}^+ = \mathrm{diag}(\underbrace{1, \ldots, 1}_{p}, \underbrace{-1, \ldots, -1}_{q}) \qquad \text{and} \qquad \mathcal{G}^- = \mathrm{diag}(\underbrace{1, \ldots, 1}_{N/2}) \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.25}$$

where $p + q = N$ and the second case is valid only for N even. Then a well-known result in linear algebra is that K is congruent over the reals to G^η, i.e. $\mathcal{U} K \mathcal{U}^t = \mathcal{G}^\eta$ for some real matrix $\mathcal{U}$. From this one sees that the $+1$-eigenspace of $\mathcal{T}$ is the Lie subalgebra $\mathfrak{so}(p, q)$ for $\eta = +1$ and $\mathfrak{sp}(N)$ for $\eta = -1$.

Once more we may now define

$$\mathfrak{gl}_N[z]^{\mathcal{T}} = \{A(z) \in \mathfrak{gl}_N[z] \mid \mathcal{T} A(z) = A(-z)\}, \tag{2.26}$$

that is, the subalgebra of $\mathfrak{gl}_n[z]$ in which each element of degree k is also in the $(-1)^k$-eigenspace of $\mathcal{T}$. The projection map $\mathfrak{gl}_N[z] \to \mathfrak{gl}_N[z]^{\mathcal{T}}$ is $ez^k \mapsto \frac{1}{2}(1 + (-1)^k \mathcal{T}) e z^k$, and, given (2.24), this sends

$$T(u) \mapsto T(u) + T(-u)^{\mathcal{T}} =: S(u), \tag{2.27}$$

which defines the formal series $S(u)$, whose expansion in inverse powers of u

$$S(u) = \frac{1}{u} S^{(0)} + \frac{1}{u^2} S^{(1)} + \cdots \tag{2.28}$$

contains a complete set of generators of $\mathfrak{gl}_N[z]^{\mathcal{T}}$. The commutation relations of this subalgebra can be written simply by using the notation with the formal series.

Lemma 2.3. *$S(u)$ obeys*

$$[S_a(u), S_b(v)] = \left[S_a(u) + S_b(v), \frac{P_{ab}}{u - v} \right] + \left[S_a(u) - S_b(v), \frac{Q_{ab}}{u + v} \right], \tag{2.29}$$

where $Q_{ab} = P_{ab}^{\mathcal{T}_a} = P_{ab}^{\mathcal{T}_b}$ and has the symmetry property that

$$S(u) = S(-u)^{\mathcal{T}}. \tag{2.30}$$

Proof. The first of these is true by virtue of (2.7) and the second is immediate from definition (2.27). □

Note that these commutation relations can be obtained from the classical limit of the twisted Yangian introduced in [15]. More abstractly, relations (2.29) and (2.30) can be regarded as defining an algebra, which can then be seen to be embedded in the half-loop algebra according to (2.27).

It is well known that the centre of this subalgebra is generated by the odd coefficients of the series $s(u) = \mathrm{tr}\, S(u)$ (see for example [21, section 4]). But we have also

Proposition 2.4. *The quantities in the expansion of* $s'(u) = \mathrm{tr}\, S(u)^2$ *are mutually commuting, or equivalently*

$$[s'(u), s'(v)] = 0, \tag{2.31}$$

for all values of u *and* v. *Moreover,*

$$[S^{(0)}, s'(u)] = 0. \tag{2.32}$$

The elements in $S^{(0)}$ *generate* $\mathfrak{so}(p,q)$ *for* $\eta = +1$ *and* $\mathfrak{sp}(N)$ *for* $\eta = -1$.

Proof. The details of the proof are given in appendix B. □

3. Gaudin models

3.1. The inner-twisted Gaudin magnets

The quantum-Gaudin magnet, introduced in [3], is an integrable spin chain with long range interactions. The Gaudin Hamiltonians for the model with L sites are

$$\mathcal{H}_k = \sum_{\substack{j=1 \\ j\neq k}}^{L} \frac{P_{jk}}{z_j - z_k}, \tag{3.1}$$

where z_i are complex numbers. (Recall that P_{jk} permutes the jth and kth spins.) This model is usually called the A_L-type Gaudin model. It may be obtained from the more general class of integrable Hamiltonians

$$H_k = \sum_{\substack{j=1 \\ j\neq k}}^{L} \frac{\mathrm{tr}_a \mathbb{P}_{ak}\mathbb{P}_{aj}}{z_k - z_j} \tag{3.2}$$

by specifying that the spin at each site j is in the fundamental representation of $\mathfrak{gl}_N$.

Now, given proposition 2.2 above, we can obtain new integrable models, as in the following proposition. These models describe spins placed at fixed positions in the plane, each of which interacts with the central point and with the other spins, not only directly, but also via their images under the rotation group of order n.

Proposition 3.1. *The model described by any one of the Hamiltonians*

$$H_k^{(n)} = \sum_{\substack{j=1 \\ j\neq k}}^{L} \sum_{p\in\mathbb{Z}_n} \frac{\mathrm{tr}_a \mathbb{P}_{ak} G_a^{-p} \mathbb{P}_{aj} G_a^{p}}{z_k - \tau^p z_j} + \sum_{p\in\mathbb{Z}_n, p\neq 0} \frac{\mathrm{tr}_a \mathbb{P}_{ak} G_a^{-p} \mathbb{P}_{ak} G_a^{p}}{2z_k} \tag{3.3}$$

is integrable. This model has $\mathfrak{gl}_{N_0} \oplus \mathfrak{gl}_{N_1} \oplus \cdots \oplus \mathfrak{gl}_{N_{n-1}}$ *symmetry.*

Proof. From definition (2.18) of $B(u)$, one finds

$$b'(u) = \mathrm{tr}\, B(u)^2 = \sum_{k=1}^{L} \sum_{j\in\mathbb{Z}_n} \frac{\tau^j}{u - \tau^{-j} z_k} H_k^{(n)} + \sum_{k=1}^{L} \sum_{j\in\mathbb{Z}_n} \frac{\mathrm{tr}_a \mathbb{P}_{ak}\mathbb{P}_{ak}}{(u - \tau^{-j} z_k)^2}, \tag{3.4}$$

with $H_k^{(n)}$ as given in the proposition. (The identity

$$\frac{1}{(u-\tau^{-j}z_l)(u-\tau^{-k}z_p)} = \frac{1}{\tau^{-j}z_l-\tau^{-k}z_p}\left(\frac{1}{u-\tau^{-j}z_l}-\frac{1}{u-\tau^{-k}z_p}\right) \tag{3.5}$$

for $(j,l)\neq(k,p)$ is helpful in showing this.)

It then follows from proposition 2.2 that $\left[H_k^{(n)}, H_p^{(n)}\right] = 0$. Since (for $n>1$) these operators $H_p^{(n)}$ are independent we have found L commuting conserved quantities, completing the proof of integrability of the Hamiltonian $H^{(n)}$. Next, from the relation $[B^{(0)}, \mathrm{tr}\, B(u)^2] = 0$, also proved in proposition 2.2, we deduce that $[B^{(0)}, H^{(n)}] = 0$, which gives the $\mathfrak{gl}_{N_0}\oplus\mathfrak{gl}_{N_1}\oplus\cdots\oplus\mathfrak{gl}_{N_{n-1}}$ symmetry of the model. □

Examples:

- For $n=2$, we obtain the Hamiltonian

$$H_k^{(2)} = \sum_{\substack{j=1\\ j\neq k}}^{L}\left(\frac{\mathrm{tr}_a\mathbb{P}_{ak}\mathbb{P}_{aj}}{z_k-z_j}+\frac{\mathrm{tr}_a\mathbb{P}_{ak}G_a\mathbb{P}_{aj}G_a}{z_k+z_j}\right)+\frac{\mathrm{tr}_a\mathbb{P}_{ak}G_a\mathbb{P}_{ak}G_a}{2z_k} \tag{3.6}$$

  of the BC-type Gaudin model studied in [19].
- If the sites carry the fundamental representation of $\mathfrak{gl}_N$, our Hamiltonian is

$$H_k^{(n)} = \sum_{\substack{j=1\\ j\neq k}}^{L}\sum_{p\in\mathbb{Z}_n}\frac{G_j^pP_{kj}G_j^{-p}}{z_k-\tau^pz_j}+\sum_{p\in\mathbb{Z}_n,\,p\neq0}\frac{G_k^{-p}\,\mathrm{tr}\,G^p}{2z_k}. \tag{3.7}$$

Let us remark that in the A_L case ($n=1$) supplementary conserved quantities, called higher Gaudin Hamiltonians, can be found by computing for example $\mathrm{tr}\,T(u)^3$ (see e.g. [22]). The question of whether this is possible in the generalized cases ($n\neq1$) studied here remains open.

3.2. The outer-twisted Gaudin magnets

Using the algebraic result of proposition 2.4, we can also succeed in constructing integrable models based on outer automorphisms, as follows.

Proposition 3.2. *The model described by any one of the Hamiltonians*

$$H_k^{\eta} = \sum_{\substack{j=1\\ j\neq k}}^{L}\left(\frac{\mathrm{tr}_a\mathbb{P}_{ak}\mathbb{P}_{aj}}{z_k-z_j}+\frac{\mathrm{tr}_a\mathbb{P}_{ak}\mathbb{Q}_{aj}}{z_k+z_j}\right)+\frac{\mathrm{tr}_a\mathbb{P}_{ak}\mathbb{Q}_{ak}}{2z_k} \tag{3.8}$$

is integrable. The model has $\mathfrak{so}(p,q)$ symmetry (resp. $\mathfrak{sp}(N)$ symmetry) for $\eta=+1$ (resp. $\eta=-1$).

Proof. The proof is similar to that of proposition 2.2. Using definition (2.27) of $S(u)$, we show that

$$s'(u) = \mathrm{tr}\,S(u)^2 = \sum_{k=1}^{L}\frac{4z_k}{(u-z_k)(u+z_k)}H_k^{\eta}+\sum_{k=1}^{L}\mathrm{tr}_a\mathbb{P}_{ak}\mathbb{P}_{ak}\left(\frac{1}{(u-z_k)^2}+\frac{1}{(u+z_k)^2}\right), \tag{3.9}$$

with H_k^{η} given as in the proposition. Then, we deduce from proposition 2.4 that $\left[H_k^{\eta}, H_p^{\eta}\right]=0$, and since the operators H_p^{η} are independent for different p, this proves the integrability of H^{η}. The symmetry algebra is deduced from $[S^{(0)}, \mathrm{tr}\,S(u)^2] = 0$ proved in proposition 2.4. □

Every choice of representation $V_1 \otimes \cdots \otimes V_L$ for the sites then yields a Gaudin-type model. (It is worth remarking that it is possible to choose different representations at different sites.) For example, in the fundamental representation of $\mathfrak{gl}_N$, the Hamiltonian is

$$H_k^\eta = \sum_{\substack{j=1 \\ j \neq k}}^{L} \left(\frac{P_{kj}}{z_k - z_j} + \frac{Q_{kj}}{z_k + z_j} \right) + \frac{\eta}{2z_k}. \tag{3.10}$$

We may interpret H^η as a Gaudin model with boundary as in the BC-type model (equation (3.6), and see also [19]). The $z_k + z_j$ term in (3.8) corresponds to the interaction between the kth spin represented in V_k and the jth 'reflected' spin transforming in the contragredient representation. This type of boundary is called soliton non-preserving and has been implemented in other integrable models [23–26]. The final term in (3.8) corresponds to the interaction between particles and the boundary.

4. Calogero models

We turn now to the second class of integrable system of interest in this work, the Calogero models. We seek to construct dynamical models of multiple particles on a star graph, whose pairwise interactions are determined by a potential of the usual Calogero type, namely $1/q^2$, where q is the linear distance separating the particles in the plane of the star graph. We will first construct models of particles of unspecified statistics; subsequently, by specifying statistics and parity, we arrive at Calogero models for particles with internal spins.

4.1. The A_L case

Let us first recall the Calogero model based on the root system A_L [4], and in particular the use of Dunkl operators [27] in demonstrating its integrability [18]. Consider a quantum-mechanical system of L particles on the real line. Let q_i be the position operator of the ith particle, and write the position-space wavefunction as

$$\psi(q_1, q_2, \ldots, q_L). \tag{4.1}$$

Let $\mathscr{P}_{ij} = \mathscr{P}_{ji}$ be the operator which transposes the positions of particles i and j,

$$\mathscr{P}_{ij}\psi(\ldots, q_i, \ldots, q_j, \ldots) = \psi(\ldots, q_j, \ldots, q_i, \ldots). \tag{4.2}$$

Let us denote by S_L the permutation group of L elements and by (ij) the transposition of the elements i and j. Each element $s \in S_L$ can be written in terms of transpositions, namely $s = (ij)\ldots(kl)$. Then, we can define $\mathscr{P}_s$ as the shorthand for the product $\mathscr{P}_{ij}\ldots\mathscr{P}_{kl}$ (even though the expression of s in terms of transpositions is not unique, $\mathscr{P}_s$ is well defined due to the commutation relations satisfied by $\mathscr{P}_{ij}$). The sign of s, denoted by $|s|$, is the number of these transpositions modulo 2.

Let us define L operators d_i—the Dunkl operators—by [28, 29]

$$d_i = p_i + \lambda \sum_{j \neq i} \frac{1}{q_i - q_j} \mathscr{P}_{ij}, \qquad \text{where} \quad p_i = -\mathrm{i}\hbar \frac{\partial}{\partial q_i}. \tag{4.3}$$

It follows from the relations $\mathscr{P}_{ij} q_j = q_i \mathscr{P}_{ij}$ that the Dunkl operators commute with one another,

$$[d_i, d_j] = 0 \tag{4.4}$$

and consequently that the quantities

$$I^{(k)} = \sum_{i=1}^{L} d_i^k \tag{4.5}$$

also form a commuting set. The $I^{(k)}$ are algebraically independent for $k = 1, 2, \ldots, L$, and these give L commuting conserved quantities of the model with Hamiltonian

$$H = I^{(2)} = \sum_{i=1}^{L} d_i^2 = \sum_{i=1}^{L} \left(p_i^2 - \sum_{j \neq i} \frac{1}{(q_i - q_j)^2} \lambda(\lambda - \mathrm{i}\hbar \mathscr{P}_{ij}) \right), \tag{4.6}$$

which is therefore, by construction, integrable.

The next step is to consider particles with internal degrees of freedom, which we take to be in the fundamental representation of $\mathfrak{gl}_N$. The wavefunction becomes

$$\psi(q_1, q_2, \ldots, q_L \mid v_1, v_2, \ldots, v_L), \tag{4.7}$$

where $v_i \in \mathbb{C}^N$. As we define operators $\mathscr{P}_{ij}$ which transpose the positions, we introduce an operator P_{ij} which transposes the spins

$$P_{ij} \psi(q_1, \ldots, q_L | \ldots, v_i, \ldots, v_j, \ldots) = \psi(q_1, \ldots, q_L | \ldots, v_j, \ldots, v_i, \ldots). \tag{4.8}$$

We define similarly to $\mathscr{P}_s$ the matrix $P_s = P_{ij} \ldots P_{kl}$ for $s = (ij) \ldots (kl)$ acting on the spins.

As explained before, to use the St Petersburg notation, we need supplementary spaces called auxiliary spaces (which are $\mathbb{C}^N$ and, in this case, isomorphic to the quantum space) and denoted by the letters $a, b, \ldots$. The conserved quantities (4.5) then emerge in a natural way from the matrix

$$T_a(u) = \sum_{\ell=1}^{L} \frac{P_{a\ell}}{u - d_\ell}, \tag{4.9}$$

because (as one can see using $\mathrm{tr}_a P_{a\ell} = 1$)

$$t(u) = \mathrm{tr}_a T_a(u) = \sum_{k=0}^{\infty} \frac{I^{(k)}}{u^{k+1}}. \tag{4.10}$$

Here (4.9) is nothing but a modified version of the monodromy matrix (2.10). The parameters z_ℓ are replaced by the Dunkl operators, and since the quantum spaces are chosen to be in the fundamental representation, $\mathbb{P}_{a\ell} = E_{ij} \otimes e_{ji}$ becomes the transposition operator $P_{a\ell} = E_{ij} \otimes E_{ji}$ (for $\ell = 1, \ldots, L$). Now because d_i commute with each other *and* with all operations on the internal degrees of freedom, $T(u)$ obeys the half-loop algebra relations (2.7) exactly as before.

Suppose, finally, that the particles are in fact indistinguishable, which is often the case of real physical interest. One must then impose definite exchange statistics on the wavefunction:

$$P_{ij} \mathscr{P}_{ij} \psi = \epsilon \psi, \tag{4.11}$$

where $\epsilon = +1$ for bosons and $\epsilon = -1$ for fermions. The projector onto such states is

$$\Lambda = \sum_{s \in S_L} \epsilon^{|s|} \mathscr{P}_s P_s. \tag{4.12}$$

The following relation

$$(\Lambda - 1) T(u) \Lambda = 0 \tag{4.13}$$

demonstrated in [18] is crucial, because it implies that the modified generators $\widetilde{T}(u) = T(u)\Lambda$ preserve the condition $\Lambda\psi = \psi$, and obey the same algebraic relations as the original $T(u)$.

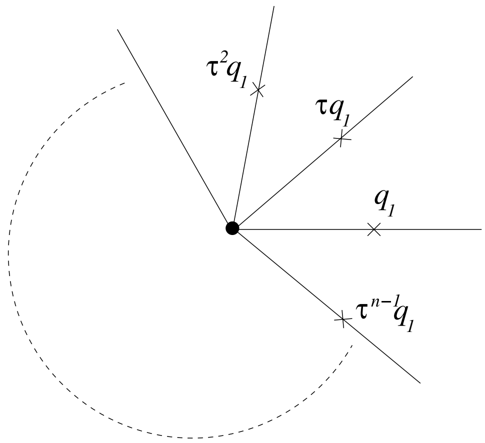

Figure 1. A particle on the branch $\mathbb{R}^+$, and its images on the other branches.

From $\widetilde{T}(u)$, we may define $\widetilde{t}(u) = \mathrm{tr}_a T(u)\Lambda = t(u)\Lambda$, and hence $\widetilde{I}^{(k)} = I^{(k)}\Lambda$. Using $[\widetilde{t}(u), \widetilde{t}(v)] = 0$, one obtains that the $\widetilde{I}^{(k)}$ are once more L commuting conserved quantities of the system with Hamiltonian

$$\widetilde{H} = \widetilde{I}^{(2)} = \sum_{i=1}^{L} d_i^2 \Lambda = \sum_{i=1}^{L} \left(p_i^2 - \sum_{j\neq i} \frac{\lambda(\lambda - \mathrm{i}\hbar\epsilon P_{ij})}{(q_i - q_j)^2} \right) \Lambda, \tag{4.14}$$

where we are now able to replace $\mathscr{P}$, which acts on particle positions, by P, which acts only on the internal degrees of freedom. Moreover, since $\widetilde{t}(u)$ commutes with $\widetilde{T}(u)$, the model has a half-loop symmetry algebra.

The subtlety in all this is that the Dunkl operators themselves do *not* obey any relation analogous to (4.13). There are thus essentially three steps in this procedure to construct an integrable Hamiltonian for a system of indistinguishable particles

(1) find commuting Dunkl operators, and hence $T(u)$,
(2) construct the appropriate projector Λ onto physical states and
(3) prove the relation $(\Lambda - 1)T(u)\Lambda = 0$.

4.2. *Dunkl operators for the order n inner-twisted case*

We can now turn to applying these ideas to the model of interest isn the present work. We consider a system of L particles living on n half lines—'branches'—joined at a central node, as in figure 1. The branches are given parametrically by $z = \tau^k t$, $t > 0$, $k \in \mathbb{Z}_n$, and we shall denote them by

$$\mathbb{R}^+, \tau\mathbb{R}^+, \ldots, \tau^{n-1}\mathbb{R}^+. \tag{4.15}$$

As before, let q_i be the position operator of the ith particle. (Note that the spectrum of q_i is not real, but only for the superficial reason that we choose to regard the half lines as subsets of the complex plane.) In addition to the $\mathscr{P}_{ij}$, which exchange particle positions, we can define now new operators $\mathscr{Q}_i$ which move the particles between branches

$$\mathscr{Q}_i \psi(\ldots, q_i, \ldots) = \psi(\ldots, \tau q_i, \ldots). \tag{4.16}$$

It is useful to collect together the algebraic relations satisfied by the q_i, $\mathscr{Q}_i$ and $\mathscr{P}_{ij}$:

$$\mathscr{P}_{ij}\mathscr{P}_{jk}\mathscr{P}_{ij} = \mathscr{P}_{jk}\mathscr{P}_{ij}\mathscr{P}_{jk}, \qquad \mathscr{P}_{ij}^2 = 1, \qquad \mathscr{P}_{ij} = \mathscr{P}_{ji}, \tag{4.17}$$

$$\mathscr{Q}_i^n = 1, \tag{4.18}$$

and

$$\mathscr{P}_{ij}\mathscr{Q}_j = \mathscr{Q}_i\mathscr{P}_{ij}, \qquad \mathscr{P}_{ij}q_j = q_i\mathscr{P}_{ij}, \qquad \tau\mathscr{Q}_i q_i = q_i\mathscr{Q}_i, \tag{4.19}$$

with all the rest commuting. To construct an integrable model, the first task is to find a suitable generalization of the commuting Dunkl operators introduced above.

Proposition 4.1. *The Dunkl operators defined by*

$$d_i = p_i + \lambda \sum_{\substack{j=1\\ j\neq i}}^{L} \sum_{k\in\mathbb{Z}_n} \frac{1}{q_i - \tau^k q_j}\mathscr{Q}_i^k\mathscr{P}_{ij}\mathscr{Q}_i^{-k} + \sum_{k\in\mathbb{Z}_n}\frac{\mu_k}{q_i}\mathscr{Q}_i^k, \qquad p_i = -\mathrm{i}\hbar\frac{\partial}{\partial q_i}, \tag{4.20}$$

for arbitrary parameters $\lambda, \mu_k \in \mathbb{C}$, commute amongst themselves:

$$[d_i, d_j] = 0. \tag{4.21}$$

Proof. Consider first the terms at order λ. We have

$$\left[\mathscr{Q}_i^k\mathscr{P}_{ij}\mathscr{Q}_i^{-k}, p_j\right] = \mathscr{P}_{ij}(\tau^k p_j - p_i)\mathscr{Q}_j^k\mathscr{Q}_i^{-k} \tag{4.22}$$

using relations (4.19) and the definition $p_j = -\mathrm{i}\hbar\frac{\partial}{\partial q_j}$, which together imply $\mathscr{Q}_j p_j = p_j\mathscr{Q}_j\tau$. The two terms of this type occurring in $[d_i, d_j]$ are

$$\sum_{k\in\mathbb{Z}_n} \frac{1}{q_i - \tau^k q_j}\left[\mathscr{Q}_i^k\mathscr{P}_{ij}\mathscr{Q}_i^{-k}, p_j\right] + \frac{1}{q_j - \tau^k q_i}\left[p_i, \mathscr{Q}_j^k\mathscr{P}_{ji}\mathscr{Q}_j^{-k}\right]$$
$$= \sum_{k\in\mathbb{Z}_n} \frac{1}{q_i - \tau^k q_j}\mathscr{P}_{ij}(\tau^k p_j - p_i)\mathscr{Q}_j^k\mathscr{Q}_i^{-k} - \frac{1}{q_j - \tau^k q_i}\mathscr{P}_{ij}(\tau^k p_i - p_j)\mathscr{Q}_i^k\mathscr{Q}_j^{-k} \tag{4.23}$$

which cancel, after a change of the summation index in the second. The two terms containing $[p_j, 1/(q_i - q_j)]$ cancel similarly. The terms occurring at order λ^2 are of the form

$$\left[\frac{1}{q_i - \tau^k q_g}\mathscr{Q}_i^k\mathscr{P}_{ig}\mathscr{Q}_i^{-k}, \frac{1}{q_j - \tau^\ell q_h}\mathscr{Q}_j^\ell\mathscr{P}_{jh}\mathscr{Q}_j^{-\ell}\right]. \tag{4.24}$$

These vanish trivially unless at least one of the indices i, g matches at least one of j, h. It is straightforward, though tedious, to check that the terms with exactly one index in common sum to zero, by using relations (4.19) to bring every such term into e.g. the form $\frac{1}{q-q}\frac{1}{q-q}\mathscr{P}\mathscr{P}\mathscr{Q}\mathscr{Q}\mathscr{Q}$ and then summing the fractions directly. The terms in which *both* indices match give

$$\sum_{k\in\mathbb{Z}_n}\mathscr{Q}_i^k\mathscr{Q}_j^{-k}\sum_{\ell\in\mathbb{Z}_n}\left(\frac{1}{\tau^{-\ell}q_j - \tau^k q_i}\frac{1}{q_j - \tau^\ell q_i} - \frac{1}{\tau^{k-\ell}q_i - \tau^{-k}q_j}\frac{1}{q_i - \tau^{\ell-k}q_j}\right), \tag{4.25}$$

and here the sum over ℓ may be re-written as

$$\frac{1}{q_i - \tau^k q_i}\sum_{\ell\in\mathbb{Z}_n}\left(\frac{1}{q_j - \tau^\ell q_i} - \frac{1}{q_j - \tau^{k+\ell}q_i} - \frac{1}{\tau^{-k}q_j - \tau^\ell q_i} + \frac{1}{\tau^{-k}q_j - \tau^{k+\ell}q_i}\right), \tag{4.26}$$

which then vanishes by shifting the dummy index ℓ in the second and fourth terms. The terms involving μ_k may be treated similarly. □

These Dunkl operators have been introduced previously in [30] as Dunkl operators associated with complex reflection groups. A proof of their commutativity is already given but is based on different computations.

As in the A_L case above, the quantities

$$I^{(k)} = \sum_{i=1}^{L} d_i^k \tag{4.27}$$

are then mutually commuting, forming a hierarchy of Hamiltonians of an integrable system. Their detailed forms are rather complicated, for example, in the case of $n = 3$ branches with only $L = 2$ particles and $\mu_k = 0$, we find that the first three are

$$I^{(1)} = p_1 + p_2 + \lambda \left(\frac{1-\tau}{q_1 - \tau q_2} \mathcal{Q}_1 \mathcal{Q}_2^{-1} + \frac{1-\tau^2}{q_1 - \tau^2 q_2} \mathcal{Q}_1^{-1} \mathcal{Q}_2 \right) \mathcal{P}_{12} \tag{4.28}$$

$$\begin{aligned} I^{(2)} = {} & p_1^2 + p_2^2 + \lambda p_1 \left(\frac{1-\tau^2}{q_1 - \tau q_2} \mathcal{Q}_1 \mathcal{Q}_2^{-1} + \frac{1-\tau}{q_1 - \tau^2 q_2} \mathcal{Q}_1^{-1} \mathcal{Q}_2 \right) \mathcal{P}_{12} \\ & + \lambda p_2 \left(\frac{1-\tau^2}{q_2 - \tau q_1} \mathcal{Q}_2 \mathcal{Q}_1^{-1} + \frac{1-\tau}{q_2 - \tau^2 q_1} \mathcal{Q}_2^{-1} \mathcal{Q}_1 \right) \mathcal{P}_{12} \\ & - 3\lambda^2 \frac{q_1^4 + 2q_1^3 q_2 + 2q_1 q_2^3 + q_2^4}{\left(q_1^3 - q_2^3\right)^2} \end{aligned} \tag{4.29}$$

$$\begin{aligned} I^{(3)} = {} & p_1^3 + p_2^3 - 3\lambda p_1 \left(\frac{1}{(q_1 - q_2)^2} (\mathcal{P}_{12} + \lambda) + \frac{1}{(q_1 - \tau q_2)^2} \left(\mathcal{P}_{12} \mathcal{Q}_1^{-1} \mathcal{Q}_2 + \lambda \right) \right. \\ & \left. + \frac{1}{(q_1 - \tau^2 q_2)^2} (\mathcal{P}_{12} \mathcal{Q}_1 \mathcal{Q}_2^{-1} + \lambda) \right) \\ & - 3\lambda p_2 \left(\frac{1}{(q_1 - q_2)^2} (\mathcal{P}_{12} + \lambda) + \frac{1}{(q_2 - \tau q_1)^2} \left(\mathcal{P}_{12} \mathcal{Q}_2^{-1} \mathcal{Q}_1 + \lambda \right) \right. \\ & \left. + \frac{1}{(q_2 - \tau^2 q_1)^2} \left(\mathcal{P}_{12} \mathcal{Q}_2 \mathcal{Q}_1^{-1} + \lambda \right) \right) \\ & - 3\sqrt{3} i \lambda^2 \frac{1}{(q_1 - q_2)(q_1 - \tau q_2)(q_1 - \tau^2 q_2)} \mathcal{Q}_1 \mathcal{Q}_2 (\mathcal{Q}_1 - \mathcal{Q}_2). \end{aligned} \tag{4.30}$$

4.3. Quasi-parity and particles with spin

The next steps are to consider particles with spin and to choose a suitable projector onto physical states. We take the latter to be the product of two parts

$$\Lambda_P \Lambda_Q, \tag{4.31}$$

where

$$\Lambda_P = \sum_{s \in S_L} \epsilon^{|s|} \mathcal{P}_s P_s \tag{4.32}$$

is the projector onto states of definite exchange statistics, and

$$\Lambda_Q = \prod_{i=1}^{L} \frac{1}{n} \sum_{j \in \mathbb{Z}_n} \mathcal{Q}_i^j G_i^{-j} \tag{4.33}$$

is a new projector which relates the wavefunction on different branches: $G_i \psi = \mathcal{Q}_i \psi$. In the case of $n = 2$, this means relating the wavefunction at $q_i = x$ with $q_i = -x$, or, in other words, imposing a parity condition, so we shall call analogous conditions for arbitrary n 'quasi-parity' conditions. The reason for requiring quasi-parity is that we would like to find a Hamiltonian that does not involve $\mathscr{P}_{ij}$ or $\mathscr{Q}_j$, and, just as imposing definite statistics allowed $\mathscr{P}$ to be replaced by P in (4.14), so here quasi-parity will allow $\mathscr{Q}$ to be replaced by G, the matrix defining the automorphism of $\mathfrak{gl}_N$.[1]

Now $T_a(u)$, defined in (4.9), does not respect quasi-parity, but we have instead

Lemma 4.2.

$$B_a(u) = \sum_{j\in\mathbb{Z}_n} \tau^j G_a^j T_a(u\tau^j) G_a^{-j} \tag{4.34}$$

satisfies

$$[B_a(u), \Lambda_Q] = 0. \tag{4.35}$$

Proof. By direct computation. □

Thus, using also that $T(u)$ obeys (4.13), we have

$$(1 - \Lambda_P\Lambda_Q)B_a(u)\Lambda_P\Lambda_Q = 0, \tag{4.36}$$

and have therefore arrived at the following result

Proposition 4.3. *The modified generators $\widetilde{B}(u) = B(u)\Lambda_P\Lambda_Q$ preserve the statistics and quasi-parity of the wavefunction and themselves satisfy relations (2.20). The quantities $\widetilde{I}^{(k)}$ in the expansion of*

$$\widetilde{b}(u) = \mathrm{tr}_a B_a(u)\Lambda_P\Lambda_Q =: \sum_{k=0}^{\infty} \frac{\widetilde{I}^{(k)}}{u^{k+1}} \tag{4.37}$$

are mutually commuting, and non-zero only when $k \equiv 0 \bmod n$. The model with this hierarchy of integrable Hamiltonians has symmetry $\mathfrak{gl}_N[z]^\sigma$.

Proof. Most of this follows from the construction above: it remains only to show that $\widetilde{I}^{(k)}$ vanishes for $k \not\equiv 0 \bmod n$. One sees this by writing

$$\widetilde{b}(u) = \sum_{\ell=1}^{L}\sum_{p=0}^{\infty}\left(\sum_{k\in\mathbb{Z}_n}\tau^{-pk}\right)\frac{d_l^p}{u^{p+1}}\Lambda_P\Lambda_Q \tag{4.38}$$

and noting that $\sum_{k\in\mathbb{Z}_n}\tau^{-pk}$ is zero unless $p \equiv 0 \bmod n$. □

The vanishing of some of the generators is as expected: in the case $n = 2$, for example, the charge $I^{(1)}$, which is of first order in momentum p, does not survive the introduction of a

[1] Note that in fact there is another natural class of quasi-parity condition: when $n = 2, \psi(\ldots, q_i, \ldots) = G_i\psi(\ldots, -q_i, \ldots)$ is obviously equivalent to $\psi(\ldots, q_i, \ldots) - G_i\psi(\ldots, -q_i, \ldots) = 0$, but these two formulations suggest different generalizations to $n > 2$: we can on the one hand demand for all i that

$$\psi(\ldots, q_i, \ldots) = G_i\psi(\ldots, \tau q_i, \ldots) = \cdots = G_i^{n-1}\psi(\ldots, \tau^{n-1}q_i, \ldots)$$

or, alternatively, for all i

$$\psi(\ldots, q_i, \ldots) + \tau G_i\psi(\ldots, \tau q_i, \ldots) + \cdots + \tau^{n-1}G_i^{n-1}\psi(\ldots, \tau^{n-1}q_i, \ldots) = 0.$$

We use the first type of quasi-parity here. With such a condition in force one need only give the wavefunction on $\mathbb{R}^+$ in order to completely specify the state, and in this sense the model is really on the half-line. The second type of quasi-parity is weaker—and so potentially interesting—but does not allow us to replace $\mathscr{Q}$ by G in the Hamiltonian.

boundary. In the case of $n = 3$ with $L = 2$ particles, already mentioned in (4.28), the first non-vanishing charge is of third order in momentum:

$$\begin{aligned}\widetilde{I}^{(3)} = p_1^3 + p_2^3 - 3\lambda p_1 \bigg(& \frac{1}{(q_1-q_2)^2}(\epsilon P_{12}+\lambda) + \frac{1}{(q_1-\tau q_2)^2}\left(\epsilon P_{12}G_1^{-1}G_2+\lambda\right) \\ & + \frac{1}{(q_1-\tau^2 q_2)^2}\left(\epsilon P_{12}G_1G_2^{-1}+\lambda\right)\bigg) \\ & - 3\lambda p_2 \bigg(\frac{1}{(q_1-q_2)^2}(\epsilon P_{12}+\lambda) + \frac{1}{(q_2-\tau q_1)^2}\left(\epsilon P_{12}G_2^{-1}G_1+\lambda\right) \\ & + \frac{1}{(q_2-\tau^2 q_1)^2}\left(\epsilon P_{12}G_2G_1^{-1}+\lambda\right)\bigg) \\ & - 3\sqrt{3}\mathrm{i}\lambda^2 \frac{1}{(q_1-q_2)(q_1-\tau q_2)(q_1-\tau^2 q_2)} G_1G_2(G_1-G_2). \end{aligned} \tag{4.39}$$

5. Conclusion

In this paper, we gave the St Petersburg presentation of subalgebras of the $\mathfrak{gl}_N$ half-loop algebras associated with all finite-order automorphisms of $\mathfrak{gl}_N$. This presentation allows us to obtain commuting quantities used to prove integrability for new integrable models of Gaudin or Calogero type. The non-Abelian symmetry for each of these new models is also exhibited.

We may expect that the usual methods to solve the Gaudin models and the Calogero models may be generalized to solve the models given in propositions 3.1, 3.2 or 4.3 introduced in this paper. Namely, for the Gaudin models, the Hamiltonians (3.3) and (3.8) may be diagonalized by generalizing the usual methods such as the separation of variables [31] or the algebraic Bethe ansatz [32]. For the Calogero models, the previous link established in [30] between the nonsymmetric Jack polynomials and the Dunkl operators (4.20) may be useful to diagonalize the Hamiltonian given in proposition 4.3.

Our discussion has dealt exclusively with quantum-mechanical models. However, for each algebra introduced in the paper, there exists an associated Poisson bracket algebra, obtained simply by replacing the commutator on the left of the defining relations (2.1), (2.3), (2.7), (2.20) and (2.29) by a Poisson bracket. This allows us to treat certain classical mechanical problems. In such problems, the entries of $T(u)$, $B(u)$ or $S(u)$ are commuting functions on phase space, which simplifies many computations. For example, the results of propositions 2.2 and 2.4 are replaced by the stronger statements

$$\{b_k(u), b_\ell(v)\} = 0 \qquad \text{and} \qquad \{s_k(u), s_\ell(v)\} = 0, \tag{5.1}$$

where $b_k(u) = \operatorname{tr} B(u)^k$ and $s_k(u) = \operatorname{tr} S(u)^k$. These results strongly suggest that the classical counterpart of models given by (3.3) and (3.8) are integrable in the sense of Liouville. (It remains to prove that the quantities are independent.)

Finally, although the models of this paper were quantum-mechanical, the *algebras* are classical, in the sense that they are not q-deformed. A very interesting question is whether a similar construction of subalgebras from higher order automorphisms of $\mathfrak{gl}_N$ is possible in the case of quantum groups. If so, then these subalgebras would be a q-deformation of those in this paper, and should also have associated with them integrable models on n half lines.

Acknowledgments

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Appendix A. Proof of proposition 2.2

It is convenient first to use

$$\begin{aligned}G_a^{-k}P_{ab}G_a^kB_a(u) &= G_a^{-k}G_b^kB_b(u)P_{ab} = G_b^kB_b(u)\big(G_b^{-k}G_b^k\big)G_a^{-k}P_{ab}\\ &= G_b^kB_b(u)G_b^{-k}G_a^{-k}P_{ab}G_a^k,\end{aligned}$$

and similar manipulations, to re-write the commutation relations (2.20) as

$$[B_a(u), B_b(v)] = \sum_{k\in\mathbb{Z}_n}(B_b(v) + \tau^kB_a(u) - \tau^kB_a(\tau^kv) - B_b(\tau^{-k}u))\frac{G_a^{-k}P_{ab}G_a^k}{u-\tau^kv}. \tag{A.1}$$

The goal here, and in the following, is to bring every term containing P_{ab} into the form $B_aB_bG_a^{-k}P_{ab}G_a^k$. Next, we have

$$\begin{aligned}[B_a(u), B_b(v)^2] &= B_b(v)[B_a(u), B_b(v)] + [B_a(u), B_b(v)]B_b(v)\\ &= \sum_{k\in\mathbb{Z}_n}\{B_b(v)(B_b(v) + \tau^kB_a(u) - \tau^kB_a(\tau^kv) - B_b(\tau^{-k}u))\\ &\quad + (B_b(v) + \tau^kB_a(u) - \tau^kB_a(\tau^kv) - B_b(\tau^{-k}u))\tau^kB_a(\tau^kv)\}\frac{G_a^{-k}P_{ab}G_a^k}{u-\tau^kv}\\ &= \sum_{k\in\mathbb{Z}_n}\{B_b(v)^2 - B_b(v)B_b(\tau^{-k}u) + \tau^{2k}B_a(u)B_a(\tau^kv) - \tau^{2k}B_a(\tau^kv)^2\\ &\quad - \tau^kB_a(\tau^kv)B_b(\tau^{-k}u) + \tau^kB_a(u)B_b(v)\\ &\quad + \tau^k[B_a(\tau^kv), B_b(\tau^{-k}u)] - \tau^k[B_a(u), B_b(v)]\}\frac{G_a^{-k}P_{ab}G_a^k}{u-\tau^kv},\end{aligned} \tag{A.2}$$

and then the brackets in the final line may be evaluated by using (A.1) once more, to give, after some manipulation of the summation indices

$$\begin{aligned}[B_a(u), B_b(v)^2] &= \sum_{k\in\mathbb{Z}_n}\{B_b(v)^2 - B_b(v)B_b(\tau^{-k}u) + \tau^{2k}B_a(u)B_a(\tau^kv) - \tau^{2k}B_a(\tau^kv)^2\\ &\quad - \tau^kB_a(\tau^kv)B_b(\tau^{-k}u) + \tau^kB_a(u)B_b(v)\}\frac{G_a^{-k}P_{ab}G_a^k}{u-\tau^kv}\\ &\quad - \sum_{j,k\in\mathbb{Z}_n}\{\tau^{2j}B_a(\tau^jv) - \tau^{2j}B_a(\tau^{j-k}u) - \tau^kB_b(\tau^{k-j}v)\\ &\quad + \tau^kB_b(v) + \tau^{j+k}B_a(u) - \tau^{j+k}B_a(\tau^jv)\}\frac{G_a^{k-j}G_b^{j-k}}{(u-\tau^kv)(u-\tau^jv)}.\end{aligned} \tag{A.3}$$

Therefore,

$$\begin{aligned}[B_a(u)^2, B_b(v)^2] &= B_a(u)[B_a(u), B_b(v)^2] + [B_a(u), B_b(v)^2]B_a(u)\\ &= \sum_{k\in\mathbb{Z}_n}B_a(u)\{B_b(v)^2 - B_b(v)B_b(\tau^{-k}u) + \tau^{2k}B_a(u)B_a(\tau^kv)\\ &\quad - \tau^{2k}B_a(\tau^kv)^2 - \tau^kB_a(\tau^kv)B_b(\tau^{-k}u) + \tau^kB_a(u)B_b(v)\}\frac{G_a^{-k}P_{ab}G_a^k}{u-\tau^kv}\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k\in\mathbb{Z}_n}\{B_b(v)^2 - B_b(v)B_b(\tau^{-k}u) + \tau^{2k}B_a(u)B_a(\tau^k v) - \tau^{2k}B_a(\tau^k v)^2\\
&- \tau^k B_a(\tau^k v)B_b(\tau^{-k}u) + \tau^k B_a(u)B_b(v)\}\tau^{-k}B_b(\tau^{-k}u)\frac{G_a^{-k}P_{ab}G_a^k}{u-\tau^k v}\\
&- \sum_{j,k\in\mathbb{Z}_n} B_a(u)\{\tau^{2j}B_a(\tau^j v) - \tau^{2j}B_a(\tau^{j-k}u) - \tau^k B_b(\tau^{k-j}v)\\
&+ \tau^k B_b(v) + \tau^{j+k}B_a(u) - \tau^{j+k}B_a(\tau^j v)\}\frac{G_a^{k-j}G_b^{j-k}}{(u-\tau^k v)(u-\tau^j v)}\\
&- \sum_{j,k\in\mathbb{Z}_n}\{\tau^{2j}B_a(\tau^j v) - \tau^{2j}B_a(\tau^{j-k}u) - \tau^k B_b(\tau^{k-j}v)\\
&+ \tau^k B_b(v) + \tau^{j+k}B_a(u) - \tau^{j+k}B_a(\tau^j v)\}\frac{G_a^{k-j}G_b^{j-k}}{(u-\tau^k v)(u-\tau^j v)}B_a(u)
\end{aligned}
\tag{A.4}
$$

In the final line, we could again add commutators to move all the B_a's to the left of the B_b's, but in fact it is not necessary to do so in order to evaluate

$$
[b'(u), b'(v)] = \mathrm{tr}_{ab}[B_a(u)^2, B_b(v)^2]. \tag{A.5}
$$

Consider first those terms containing P_{ab}. Because we avoided terms of the type $B_bB_aB_bP_{ab}$ in (A.4), these all reduce to single traces: for example

$$
\begin{aligned}
\mathrm{tr}_{ab}B_a(u)B_b(v)^2G_a^{-k}P_{ab}G_a^k &= \mathrm{tr}_{ab}G_a^kB_a(u)G_a^{-k}B_b(v)^2P_{ab}\\
&= \mathrm{tr}_{ab}\tau^{-k}B_a(\tau^{-k}u)P_{ab}B_a(v)^2 = \mathrm{tr}\,\tau^{-k}B_a(\tau^{-k}u)B_a(v)^2,
\end{aligned}
$$

where the final equality is valid because $\mathrm{tr}_bP_{ab} = 1$. The remaining terms give products of traces, and, after some cancellation, one finds

$$
\begin{aligned}
[b'(u), b'(v)] = &\sum_{k\in\mathbb{Z}_n}\frac{2\tau^{-k}}{u-\tau^k v}\mathrm{tr}[B(\tau^{-k}u)^2, B(v)]\\
&+ \sum_{j,k\in\mathbb{Z}_n}\{\mathrm{tr}\,\tau^j B(u)B(\tau^j v)G^{k-j}\mathrm{tr}\,G^{j-k} + \tau^k\mathrm{tr}\,B(\tau^k v)B(u)G^{k-j}\mathrm{tr}\,G^{j-k}\\
&- \mathrm{tr}\,B(u)G^{k-j}\mathrm{tr}\,B(v)G^{j-k} - \mathrm{tr}\,B(v)G^{j-k}\mathrm{tr}\,B(u)G^{k-j}\}\frac{\tau^k-\tau^j}{(u-\tau^k v)(u-\tau^j v)}.
\end{aligned}
\tag{A.6}
$$

The first term, cubic in B, reduces, given the identity $\mathrm{tr}[M, N] = \mathrm{tr}_{ab}[M_a, N_b]P_{ab}$ and (A.3), to

$$
\begin{aligned}
&\sum_{j,k\in\mathbb{Z}_n}\{\mathrm{tr}\,B(\tau^j v)B(u)G^{j-k}\mathrm{tr}\,G^{k-j} - \mathrm{tr}\,B(u)B(\tau^j v)G^{k-j}\mathrm{tr}\,G^{j-k}\\
&\quad - \tau^{-j}\mathrm{tr}\,B(u)G^{j-k}\mathrm{tr}\,B(v)G^{k-j} + \tau^{-j}\mathrm{tr}\,B(v)G^{j-k}\mathrm{tr}\,B(u)G^{k-j}\}\\
&\quad\times\frac{2\tau^{j+k}}{(u-\tau^k v)(u-\tau^j v)},
\end{aligned}
\tag{A.7}
$$

and, on collecting terms, one has

$$
\begin{aligned}
[b'(u), b'(v)] = &\sum_{j,k\in\mathbb{Z}_n}\{\tau^j\mathrm{tr}\,[B(\tau^j v)G^{k-j}, B(u)]\,\mathrm{tr}\,G^{j-k}\\
&+[\mathrm{tr}\,B(v)G^{j-k}, \mathrm{tr}\,B(u)G^{k-j}]\}\frac{\tau^j+\tau^k}{(u-\tau^j v)(u-\tau^k v)}.
\end{aligned}
\tag{A.8}
$$

The second commutator can be shown to vanish, and on evaluating the first one is left with

$$[b'(u), b'(v)] = \sum_{j,k,l\in\mathbb{Z}_n} \{\operatorname{tr} B(\tau^{-l}u)G^{k-l} \operatorname{tr} G^{l-j} - \operatorname{tr} B(\tau^{-l}u)G^{l-j} \operatorname{tr} G^{k-l} + \operatorname{tr} B(v)G^{l-j} \operatorname{tr} G^{k-l} - \operatorname{tr} B(v)G^{k-l} \operatorname{tr} G^{l-j}\} \frac{(\tau^j + \tau^k) \operatorname{tr} G^{j-k}}{(u-\tau^j v)(u-\tau^k v)(u-\tau^l v)}. \tag{A.9}$$

Consider now the two terms containing $B(v)$. After taking $\frac{1}{3}$, the sum over the cyclic permutations of the dummy indices i, j, k, one finds that these reduce to

$$\sum_{j,k,l\in\mathbb{Z}_n} \operatorname{tr} B(v)G^{j-k} \operatorname{tr} G^{k-l} \operatorname{tr} G^{l-j} \frac{\tau^k - \tau^j}{(u-\tau^j v)(u-\tau^k v)(u-\tau^l v)} \tag{A.10}$$

and the coefficient of $\operatorname{tr} B(v)G^{-a-b} \operatorname{tr} G^a \operatorname{tr} G^b$ in this sum is (with a factor $\frac{1}{2}$ when $a = b$)

$$\sum_{l\in\mathbb{Z}_n} \frac{\tau^{a+l} - \tau^{b-l}}{(u-\tau^{a+l}v)(u-\tau^{l-b}v)(u-\tau^l v)} + \frac{\tau^{b+l} - \tau^{a-l}}{(u-\tau^{b+l}v)(u-\tau^{l-a}v)(u-\tau^l v)}, \tag{A.11}$$

which may be seen to vanish by using

$$\frac{\tau^{a+l} - \tau^{l-b}}{(u-\tau^{a+l}v)(u-\tau^{l-b}v)} = \frac{1/v}{u-\tau^{a+l}v} - \frac{1/v}{u-\tau^{l-b}}$$

and the same identity with $(a \leftrightarrow b)$. Similar arguments hold for the $B(u)$ terms in (A.9), and we have, finally, that

$$[b'(u), b'(v)] = 0. \tag{A.12}$$

It remains to show that $[B^{(0)}, b'(u)] = 0$. This may be seen by expanding (A.3) to leading order in $1/u$ and taking the trace in space b. The elements in $B^{(0)}$ are the +1-eigenspace of σ, therefore they generate $\mathfrak{gl}_{N_0} \oplus \mathfrak{gl}_{N_1} \oplus \cdots \oplus \mathfrak{gl}_{N_{n-1}}$.

Appendix B. Proof of proposition 2.4

Let us rewrite relation (2.29) as

$$[S_a(u), S_b(v)] = (S_a(u) + S_b(v) - S_b(u) - S_a(v))\frac{P_{ab}}{u-v} + \left[S_a(u) - S_b(v), \frac{Q_{ab}}{u+v}\right]. \tag{B.1}$$

Note that, in contrast to the previous case computed in appendix A, here the Q cannot be moved through the S. We are now in the position to compute the bracket $[S_a(u), S_b(v)^2]$ by again using (B.1) to bring every term containing P_{ab} on the right-hand side into the form $S_a S_b P_{ab}$. We find

$$[S_a(u), S_b(v)^2] = (S_a(u)(S_a(v) + S_b(v)) - S_a(v)^2 - (S_a(v) + S_b(v))S_b(u) + S_b(v)^2)\frac{P_{ab}}{u-v} + S_a(u)\frac{Q_{ab}}{u+v}S_b(v) - S_b(v)\frac{Q_{ab}}{u+v}S_a(u) + \left[(S_a(u) - S_b(v))S_b(v), \frac{Q_{ab}}{u+v}\right] + \left[S_a(v) - S_b(u) - S_a(u) + S_b(v), \frac{\eta Q_{ab}}{u^2 - v^2}\right], \tag{B.2}$$

where we also used the property $P_{ab}Q_{ab} = \eta Q_{ab} = Q_{ab}P_{ab}$. Now we can compute $[S_a(u)^2, S_b(v)^2] = S_a(u)[S_a(u), S_b(v)^2] + [S_a(u), S_b(v)^2]S_a(u)$ and take the trace in spaces a and b. It is then straightforward to show that

$$[s'(u), s'(v)] = \frac{2}{u-v}\operatorname{tr}[S(u)^2, S(v)] + \frac{1}{u+v}(\operatorname{tr}[S(u)^2, S(-v)] - \operatorname{tr}[S(-u)^2, S(v)]), \tag{B.3}$$

where we have used the symmetry relation (2.30) and, for example, the following properties:

$$\mathrm{tr}_{ab} S_a(u) Q_{ab} S_b(u) = \mathrm{tr}_{ab} S_a(u) Q_{ab} S_a(u)^{\mathcal{T}} = \mathrm{tr}\, S(u) S(u)^{\mathcal{T}} \tag{B.4}$$

$$\mathrm{tr}_{ab} Q_{ab} S_a(u) S_b(v) S_a(u) = \mathrm{tr}_{ab} S_b(u)^{\mathcal{T}} S_b(v) S_b(u)^{\mathcal{T}} Q_{ab} = \mathrm{tr}\, S(u)^{\mathcal{T}} S(v) S(u)^{\mathcal{T}}. \tag{B.5}$$

Next, using the property

$$\mathrm{tr}[S(x)^2, S(y)] = -\mathrm{tr}[S(y), S(x)^2] = -\mathrm{tr}[S_a(y), S_b(x)^2] P_{ab} \tag{B.6}$$

and relation (B.2), we have that $\mathrm{tr}[S(x)^2, S(y)] = \frac{N}{x-y}\,\mathrm{tr}[S(y), S(x)] = \frac{N}{x-y}\,\mathrm{tr}[S_a(y), S_b(x)] P_{ab}$. Then using relation (2.29), we get $\mathrm{tr}[S(x)^2, S(y)] = 0$ which implies that the RHS of (B.3) vanishes and proves that $[s'(u), s'(v)] = 0$. Finally, expanding (B.2) to first order in $1/u$ and taking the trace in space b yields $[S^{(0)}, \mathrm{tr}\, S^2] = 0$. The elements in $S^{(0)}$ are the $+1$-eigenspace of $\mathcal{T}$, therefore they generate $\mathfrak{so}(p, q)$ for $\eta = +1$ and $\mathfrak{sp}(N)$ for $\eta = -1$. This completes the proof of proposition 2.4.

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